UDC 62-50

ON A CLASS OF DIFFERENTIAL GAMES WITH AN INTEGRAL CONSTRAINT

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Some method of constructing a u-stable bridge is described. A class of games with an integral constraint is indicated, in application to which this method permits the construction of the u-stable bridge in an explicit form. Following the scheme presented in [1-3], the first player's strategy extremal to the u-stable bridge can be constructed, ensuring that the game's position hits the terminal set.

1. The motion of vector z in a k-dimensional Euclidean space R^k is subject to the equation

$$z^{\cdot} = Cz + Nu + v, \quad u \in R^l, \quad v \in Q \tag{1.1}$$

Here C is a constant $k \times k$-matrix, R^l is an l-dimensional Euclidean space, N is a constant $k \times l$-matrix, Q is a convex compactum in R^k. The integral constraint

$$\mu(t) = \mu_0 - \int_0^t |u(\tau)|^2 d\tau \geqslant 0 \tag{1.2}$$

is imposed on the choice of control u. Here $|u|$ is the Euclidean norm of vector u and μ_0 is some positive constant. The set of numbers $\mu \geqslant 0$ is denoted by I. Then $R^k \times I$ implies the direct product of R^k and I and the game position is a point $[z, \mu]$ from $R^k \times I$. An m-dimensional Euclidean space R^m $(m \leqslant k)$ and a linear mapping π of space R^k into R^m are assumed given.

A terminal set Z is singled out in $R^k \times I$, having the form:

$$Z = \{[z, \mu]: \pi z = 0, \ \mu \geqslant 0\} \tag{1.3}$$

In order to formulate a u-stability condition [1-3] in a form convenient later on, following [4] we introduce a multiple-valued mapping $T_\sigma(X)$.

Let X be some closed set in $R^k \times I$ and let $\sigma \geqslant 0$. Then $T_\sigma(X)$ is the set of positions $[z_0, \mu_0]$ for each of which we can find, from any control $v(t) \in Q$ measurable on the interval $[0, \sigma]$, a measurable control $u(t)$ satisfying constraint (1.2) for $0 \leqslant t \leqslant \sigma$, such that $[z(\sigma), \mu(\sigma)] \in X$. Here $[z(\sigma), \mu(\sigma)]$ is the game's position at the instant σ.

For a given number $t_1 > 0$ we are required to construct a family of non-empty closed sets $W(t) \subset R^k \times I$, defined for $0 \leqslant t \leqslant t_1$ and satisfying the conditions $W(0) \subset Z$ and

$$W(t) \subset T_\sigma(W(t-\sigma)) \quad \text{for} \quad 0 < \sigma < t \leqslant t_1 \tag{1.4}$$

Mapping T_σ possesses a number of properties [4]. The following properties will be used in Sect. 2.

Property 1. $T_{\sigma_1}(T_{\sigma_2}(X)) \subset T_{\sigma_1+\sigma_2}(X)$

Property 2. $T_\sigma(X) \subset T_\sigma(X_1)$ when $X \subset X_1$

Property 3. If X is closed and $T_\sigma(X) \neq \varnothing$, then $T_\sigma(X)$ is closed.

2. We describe one method for constructing a u-stable bridge. Let a closed set Y be given in a q-dimensional linear normed space R^q, with the norm $\| y \|$, $y \in R^q$. We assume that a nonempty closed set $B(t, y) \subset R^k \times I$ has been defined for each $y \in Y$ and $t \geqslant 0$.

Condition A. If the sequence of vectors $y_n \in Y$ converges to vector $y \in Y$ and the point $[z, \mu]$ belongs to set $B(t, y)$, then there exists a sequence of points $[z_n, \mu_n] \in B(t, y_n)$ which converges to point $[z, \mu]$.

Let a number $\varepsilon > 0$ be given and let a function $f(\sigma, t, y)$ with values in set Y be defined for any $y \in Y$, $t \geqslant 0$, $0 < \sigma \leqslant \varepsilon$.

Condition B. The inclusion $T_\sigma(B(t, y)) \supset B(t + \sigma, f(\sigma, t, y))$ is fulfilled for any $y \in Y$, $t \geqslant 0$, $0 < \sigma \leqslant \varepsilon$.

Condition C. A continuous q-dimensional vector function $F(t, y)$ is defined for all $y \in Y$ and $t \geqslant 0$, such that the equality

$$\lim_{i\to\infty} (f(\sigma_i, t_i, y_i) - y_i) / \sigma_i = F(t, y) \tag{2.1}$$

is fulfilled for any sequences of $y_i \in Y$, $t_i \geqslant 0$ and $0 < \sigma_i \leqslant \varepsilon$ converging to y, t and 0, respectively.

Theorem 1. Through the point $y_0 \in Y$ let there pass the solution $y(t) \in Y$, unique on the interval $[0, t_1]$, of the Cauchy problem

$$y^{\cdot} = F(t, y), \quad y(0) = y_0 \tag{2.2}$$

Then the family of sets $W(t) = B(t, y(t))$ satisfies inclusion (1.4).

Note. The uniqueness of the solution $y(t)$ for $0 \leqslant t \leqslant t_1$ is understood in the sense that if $y_1(t) \in Y$ is a solution of problem (2.2) for $0 \leqslant t \leqslant t_2$ and $t_2 < t_1$, then $y_1(t) = y(t)$ for $0 \leqslant t \leqslant t_2$.

Proof of the theorem. We fix a number $\gamma > 0$ and we consider the closed bounded set

$$Y_1 = \{y \in Y\colon \| y - y(t) \| \leqslant \gamma \quad \text{for} \quad 0 \leqslant t \leqslant t_1\} \tag{2.3}$$

As follows from Condition C, a number $0 < \varepsilon_0 \leqslant \varepsilon$ exists such that

$$\| f(\sigma, t, y) - y \| \leqslant (\sigma\gamma) / \varepsilon_0 \quad \text{for} \quad y \in Y_1, \quad 0 \leqslant t \leqslant t_1 \quad (2.4)$$
$$0 < \sigma \leqslant \varepsilon_0$$

We take any numbers $0 \leqslant t_0 < t_2 \leqslant t_1$ satisfying the condition $\sigma = t_2 - t_0 \leqslant \varepsilon_0$. Let us show that

$$T_\sigma (B(t_0, y(t_0))) \supset B(t_2, y(t_2)) \quad (2.5)$$

From this it will follow that the set $W(t) = B(t, y(t))$ satisfies inclusion (1.4) for $0 \leqslant t \leqslant t_1$ and $0 < \sigma \leqslant \min(\varepsilon_0; t)$. Applying properties 1 and 2 of mapping T_σ, we can have that inclusion (1.4) is fulfilled for all $0 < \sigma < t \leqslant t_1$.

We partition interval $[t_0, t_2]$ into n equal parts of length $\sigma_n = \sigma / n$, and consider the finite collection of vectors

$$y_n(0) = y(t_0), \ldots, \quad y_n(i) = f(\sigma_n, t_0 + i\sigma_n, y_n(i-1)), \quad i = 1, \ldots, n \quad (2.6)$$

As follows from Condition B and from properties 1 and 2 of mapping T_σ, the inclusion

$$T_\sigma (B(t_0, y(t_0))) \supset B(t_2, y_n(n)) \quad (2.7)$$

is fulfilled for each n. According to property 3 of mapping T_σ, the set consisting of the left-hand side of inclusion (2.7) is closed. Therefore, as follows from Condition A, to prove inclusion (2.5) it is sufficient to show that some subsequence of the sequence of vectors $y_n(n)$ converges to vector $y(t_2)$. For each n the vectors (2.6) possess the following properties

$$y_n(i) \in Y_1, \quad \| y_n(i) - y_n(i-1) \| \leqslant (\sigma_n \gamma) / \varepsilon_0, \quad i = 1, \ldots, n \quad (2.8)$$

These properties are proved by induction over i with the use of inequality (2.4) and of the definition of set (2.3).

For $t_0 \leqslant t \leqslant t_2$ we define the polygonal line

$$x_n(t) = y_n(i-1) + \frac{y_n(i) - y_n(i-1)}{\sigma_n}(t - t_0 - (i-1)\sigma_n) \quad (2.9)$$

$$(i-1)\sigma_n \leqslant t - t_0 < i\sigma_n$$

$$x_n(t) = y_n(n) \quad \text{for} \quad t = t_2$$

The function $x_n(t)$ is continuous for $t_0 \leqslant t \leqslant t_2$ and, as follows from inequality (2.8), $\| x_n^{\cdot}(t) \| \leqslant \gamma / \varepsilon_0$ for almost all $t_0 \leqslant t \leqslant t_2$. Hence it follows that function $x_n(t)$ satisfies a Lipschitz condition with constant γ / ε_0. Therefore, the sequence of functions $x_n(t)$ satisfies the hypothesis of Arzelá theorem. Therefore (passing, if necessary to a subsequence), we can reckon that the sequence $x_n(t)$ converges to some function $x(t)$. The limit function $x(t)$ also satisfies a Lipschitz condition with the same constant γ / ε_0. Therefore, its derivative exists almost everywhere for $t_0 \leqslant t \leqslant t_2$. In addition, it follows from inclusion (2.8) and from the first equality in (2.6) that

$$x(t_0) = y(t_0), \quad x(t) \in Y_1 \subset Y \quad \text{for} \quad t_0 \leqslant t \leqslant t_2 \tag{2.10}$$

Let us show that

$$x^\cdot(t) = F(t, x(t)) \quad \text{for} \quad t_0 \leqslant t \leqslant t_2 \tag{2.11}$$

Let the derivative $x^\cdot(t)$ exist at the point $t_0 \leqslant t < t_2$. The equality

$$(x(t+h) - x(t)) / h = \lim_{n\to\infty} \int_0^1 x_n^\cdot (t + h\tau)\, d\tau \tag{2.12}$$

is fulfilled for any $0 < h < t_2 - t$. From formulas (2.6) and (2.9) it follows that the equality

$$x_n^\cdot (t + h\tau) = (f(\sigma_n, t_0 + \tau_n\sigma_n, x_n(t_0 + \tau_n\sigma_n)) - x_n(t_0 + \tau_n\sigma_n)) / \sigma_n \tag{2.13}$$

is fulfilled for almost all $0 \leqslant \tau \leqslant 1$. Here τ_n denotes the integer part of number $(t + h\tau - t_0) / \sigma_n$. Since the number sequence $\tau_n\sigma_n$ converges to $t + h\tau - t_0$ as $n \to \infty$ and the sequence of functions $x_n(t)$ converges uniformly to $x(t)$, we have $\lim x_n(t_0 + \tau_n\sigma_n) = x(t + h\tau)$ as $n \to \infty$. Therefore, from equality (2.13) and Condition C it follows that for almost all $0 \leqslant \tau \leqslant 1$ the sequence of $x_n^\cdot(t + h\tau)$ converges to $F(t + h\tau, x(t + h\tau))$ as $n \to \infty$. In addition, $\| x_n^\cdot (t + h\tau) \| \leqslant \gamma / \varepsilon_0$. Consequently, applying Lebesgue's theorem [5] to equality (2.12), we obtain

$$(x(t+h) - x(t)) / h = \int_0^1 F(t + h\tau, x(t + h\tau))\, d\tau$$

Passing in the latter equality to the limit as $h \to 0$ and using the continuity of function $F(t, y)$ for $t \geqslant 0$ and $y \in Y$ and also using inclusion (2.10), we obtain (2.11). Therefore, equality (2.11) is fulfilled for almost all $t_0 \leqslant t \leqslant t_2$. From the continuity of function $F(t, y)$ it follows that it is fulfilled for all $t_0 \leqslant t \leqslant t_2$. Therefore, allowing for relation (2.10) and for the uniqueness condition of the solution of problem (2.2) when $0 \leqslant t \leqslant t_1$, we obtain the equality $x(t) = y(t)$ for all $t_0 \leqslant t \leqslant t_2$. Thus we have proved that a subsequence of the sequence of vectors $y_n(n) = x_n(t_2)$ exists converging to vector $y(t_2)$.

3. We construct a u-stable bridge $W(t)$ for the game in Sect. 1 under the following assumptions:

1°. $\pi e^{tC} Q = \alpha(t) U, \quad \alpha(t) \geqslant 0$ for $t \geqslant 0$

2°. $\{\pi e^{tC} Nu: |u| \leqslant 1\} = \beta(t) S, \quad \beta(t) \geqslant 0$ for $t \geqslant 0$

3°. $S \stackrel{*}{-} \nu U \neq \varnothing$ for $0 \leqslant \nu \leqslant 1$

Here U and S are convex compacta in R^m and S is symmetric relative to the origin and contains the null vector as an interior point; $S \stackrel{*}{-} \nu U$ is the geometric difference [6] of sets S and νU; $\alpha(t)$ and $\beta(t)$ are continuous scalar functions. We note first of all that functions $\alpha(t)$ and $\beta(t)$ can vanish only at isolated points. Otherwise it can be shown that they are identically zero.

Assumption 4°. Functions $\alpha(t)$ and $\beta(t)$ are not identically zero and $\lim[\alpha(\tau)/\beta(\tau)] = \rho(t)$ as $\tau \to t$, where $\rho(t)$ is a function continuous when $t \geqslant 0$.

We introduce the notation

$$\pi_1(t) = \pi e^{tC} \tag{3.1}$$

Then from assumptions 1° and 2° we can obtain

$$\int_0^\sigma \pi_1(t+\sigma-\tau)\, Q\, d\tau = \Big(\int_t^{t+\sigma} \alpha(\tau)\, d\tau\Big) U \tag{3.2}$$

$$\Big\{\int_0^\sigma \pi_1(t+\sigma-\tau)\, u(\tau)\, d\tau : \int_0^\sigma |u(\tau)|^2\, d\tau = p\Big\} = \Big(p \int_t^{t+\sigma} \beta^2(\tau)\, d\tau\Big)^{1/2} S \tag{3.3}$$

For each $t \geqslant 0$, $y_1 \geqslant 0$, $y_2 \geqslant 0$ and $\sigma > 0$ we set

$$B(t, y_1, y_2) = \{[z, \mu]:\ \pi_1(t) z \in y_1 (\mu^{1/2} S \overset{*}{-} y_2 U),\ \mu^{1/2} \geqslant y_2\} \tag{3.4}$$

$$f_1(\sigma, t, y_1, y_2) = \Big(y_1 y_2 + \int_t^{t+\sigma} \alpha(\tau)\, d\tau\Big) \Big/ f_2(\sigma, t, y_1, y_2) \tag{3.5}$$

$$f_2(\sigma, t, y_1, y_2) = \Big(y_2^2 + \Big[\Big(\int_t^{t+\sigma} \alpha(\tau)\, d\tau\Big)^2 \Big/ \int_t^{t+\sigma} \beta^2(\tau)\, d\tau\Big]\Big)^{1/2} \tag{3.6}$$

Lemma. $T_\sigma(B(t, y_1, y_2)) \supset B(t+\sigma, f_1(\sigma, t, y_1, y_2), f_2(\sigma, t, y_1, y_2))$.

Proof. Let a point $[z, \mu]$ belong to the set on the right-hand side of the inclusion to be proved. Then from (3.4) and (3.6) it follows that

$$\pi_1(t+\sigma) z \in f_1(\sigma, t, y_1, y_2)\ (\mu^{1/2} S \overset{*}{-} f_2(\sigma, t, y_1, y_2) U) \tag{3.7}$$

$$\mu^{1/2} \geqslant f_2(\sigma, t, y_1, y_2) > y_2 \tag{3.8}$$

From the definition of mapping T_σ, from the form of set (3.4), and also from equalities (3.1) – (3.3) it follows that the point $[z, \mu]$ belongs to set $T_\sigma(B(t, y_1, y_2))$ if the inclusion

$$\pi_1(t+\sigma) z \in [(\varepsilon_1 S \overset{*}{-} \delta_1 U) + \varepsilon_2 S] \overset{*}{-} \delta_2 U \tag{3.9}$$

is fulfilled for some $p \geqslant 0$ and $(\mu - p)^{1/2} \geqslant y_2$. Here

$$\varepsilon_1 = y_1 (\mu - p)^{1/2}, \quad \delta_1 = y_1 y_2$$
$$\varepsilon_2 = \Big(p \int_t^{t+\sigma} \beta^2(\tau)\, d\tau\Big)^{1/2}, \quad \delta_2 = \int_t^{t+\sigma} \alpha(\tau)\, d\tau \tag{3.10}$$

We now indicate a number $p \geqslant 0$ satisfying the condition $(\mu - p)^{1/2} \geqslant y_2$, for which the set on the right-hand side of inclusion (3.9) coincides with the set on the right-hand side of inclusion (3.7). We set

$$p = \mu(1 - [y_2^2 / f_2^2(\sigma, t, y_1, y_2)]) \geqslant 0 \tag{3.11}$$

Then, as follows from inequality (3.8)

$$(\mu - p)^{1/2} = (\mu^{1/2} y_2) / f_2(\sigma, t, y_1, y_2) \geqslant y_2$$

Substituting this value of p into relation (3.10) and using notation (3.6), we can obtain the equality $\varepsilon_1\delta_2 = \varepsilon_2\delta_1$. Hence it follows that $\varepsilon_1 = \varphi\varepsilon_2$ and $\delta_1 = \varphi\delta_2$ for some $\varphi \geqslant 0$. Therefore, the set on the right-hand side of inclusion (3.9) has the form

$$((\varphi(\varepsilon_2 S) \overset{*}{-} \varphi(\delta_2 U)) + \varepsilon_2 S) \overset{*}{-} \delta_2 U \tag{3.12}$$

In [4, 7], when proving the equality $T_{\sigma_1+\sigma_2} = T_{\sigma_1}T_{\sigma_2}$ for a game with a simple motion, it was shown that a set of form (3.12) equals

$$(\varphi\varepsilon_2 + \varepsilon_2) S \overset{*}{-} (\varphi\delta_2 + \delta_2) U = (\varepsilon_1 + \varepsilon_2) S \overset{*}{-} (\delta_1 + \delta_2) U \tag{3.13}$$

Substituting the values of p from (3.11) into formulas (3.10) and using notation (3.5) and (3.6), we obtain $\varepsilon_1 + \varepsilon_2 = \mu^{1/2} f_1(\sigma, t, y_1, y_2)$ and $\delta_1 + \delta_2 = f_1(\sigma, t, y_1, y_2) \cdot f_2(\sigma, t, y_1, y_2)$. Consequently, set (3.13) and, therefore, the right-hand side of inclusion (3.9), coincide with the set on the right-hand side of inclusion (3.7). Thus, Condition B from Sect. 2 is fulfilled and the vector function $f(\sigma, t, y)$ has been defined for $t \geqslant 0$, $\sigma > 0$, $y_1 \geqslant 0$ and $y_2 \geqslant 0$ by relations (3.5) and (3.6). From relations (3.5) and (3.6) and from assumption 4° we can obtain that limit (2.1) has the form

$$F_1(t, y_1, y_2) = -[\rho(t) / 2y_2^2] + \alpha(t) / y_2 \tag{3.14}$$
$$F_2(t, y_1, y_2) = \rho(t) / 2y_2$$

for $t \geqslant 0$, $y_1 \geqslant 0$ and $y_2 > 0$. These functions are not defined when $y_2 = 0$. Therefore, we fix an arbitrary number $\delta > 0$ and as the set Y for which Conditions A, B and C were formulated in Sect. 2 we consider $y_1 \geqslant 0$, $y_2 \geqslant \delta$. Then functions (3.5), (3.6) and (3.14) and the family of sets (3.4) satisfy Conditions B and C on this set Y.

By assumption set S contains the null vector as an interior point. Using this we can show that the family of sets (3.4) satisfies Condition A.

Let $y_1(t)$ and $y_2(t)$ satisfy Eq.(2.2) with right-hand side (3.14) and initial conditions $y_1(0) = 0$, $y_2(0) = \delta$. Then on the basis of Theorem 1 the family of sets $W(t)$, which is obtained from (3.4) under the substitution $y_1 = y_1(t)$ and $y_2 = y_2(t)$, satisfies inclusion (1.4). In addition, as is seen from (1.3) and (3.4), $W(0) = B(0, 0, \delta) \subset Z$. Thus, the family of sets $W(t)$ found is a u-stable bridge leading onto target (1.3).

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